Instantaneous action-at-a-distance representation of field theories

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Wave theories, a specialized class of field theories, are defined and treated. The inhomogeneous wave equation with point sources in an arbitrary state of motion, satisfied by the field components of a wave theory, is solved without reference to retarded time. The resulting representation is in the form of a Lagrange series evaluated at the present time. As such, it constitutes a relativistically correct instantaneous action-at-a-distance formulation of field solutions to wave equations with point sources. Implications of this formulation with respect to wave-particle dualism are addressed. Electromagnetic theory is recast using the Lagrange series formalism. Present-time electromagnetic-field structures are developed and shown to be equivalent to classical retarded-time formulations. Alternative structures for the electromagnetic fields and potentials are introduced and validated. In the context of electromagnetic theory, advanced potentials are also discussed, as are the possibility and necessity of experimentally determining the phenomenological mix of advanced and retarded electromagnetic potentials. Ordinary causality is examined and called into question in light of the material presented, as is the mathematical structure of the electromagnetic and other wave-theoretic potentials.

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I. INTRODUCTION

This paper presents the mathematical foundations of action-at-a-distance field theory. It sets forth some of the methods whereby certain field theories may be replaced by equivalent action-at-a-distance theories, thereby facilitating the elimination of the field portion of particle-to-particle interaction processes. It does this rigorously and completely up to the representation of fields in terms of source particle motion expressed in the present time (as opposed to retarded- or advanced-time representations). It is incomplete, however, in that it does not address the conversion of specific fields into forces. Therefore the replacement of field governing partial differential equations with ordinary differential equations governing the motion of interacting particles—one of the principal aims of this effort—must be separately considered.

Because of the fundamental nature of this undertaking, emphasis has been placed upon mathematical rigor, and proofs of all claims have been presented in full in the body of this paper rather than in the Appendix. Admittedly, this makes the reading somewhat difficult. However, in presenting proofs, it is one of the aims of this paper to display some of the techniques employed and conventions adopted, in addition to the proofs themselves. This is deemed an essential part of treating the material if further development of the subject matter is to take place.

In accord with Webster [1], this paper holds that "It is the lofty aim of mathematical or theoretical physics to describe the universe in the most accurate manner. This manner must be by means of mathematics." Thus stressing the mathematics, little in the way of physical application will be found in this paper. Rather, attention has been focused on the derivation and validation of basic formulas, with considerable emphasis placed upon the development of the electromagnetic field and potentials. What *is* of physical significance is to be found in the remainder of the Introduction and in the section on advanced potentials—and that material will be found to be of a rather fundamental nature.

Except for a reference to Rouché's theorem in Sec. III, this paper, with all its proofs, is self-contained. There are two reasons for this. First, this material opens an unexplored area of investigation in theoretical physics. Further investigation will be greatly facilitated, therefore, if all the necessary material can be located in one place. Second, while Lagrange series expansions, with which this paper deals, are known, they have not before been applied to wave theories of fields in general. Therefore further research in this area may benefit from the details of Lagrange series manipulation presented herein, most of which have not appeared elsewhere, with one exception which requires some explanation: the proof of the inversion theorem presented in Sec. III is offered in full—though it is already known—since some of its detail is used later in a consideration of Lagrange series convergence and multiple roots, and in the Lagrange series development of the electromagnetic potentials. The proof does not depend, however, upon Teixeira's extended form of Bürmann's theorem [2], but is specialized to the field-theoretic application at hand; namely, the elimination of explicit retarded- and advanced-time references from certain field representations.

Since the thrust of this paper is directed toward an action-at-a-distance representation of particle-to-particle interactions, some background material on the action-at-a-distance concept will be offered next, followed by a brief rationale for this paper's undertaking, and then by a formal introduction to the material of the paper.

A. Background

Action at a distance is a concept which refers to the ability of one object to exert force on another without direct contact of the objects. It achieved high visibility after the 1687 publication of Newton's *Principia* wherein planetary motion was satisfactorily described by Newton's action-at-a-distance representation of the law of gravity. The mechanism of force transmission was not understood at that time, however. After the introduction of the continuous field concept by Faraday (1791–1867) and its later extension to the laws of electromagnetism by Maxwell (1831–1879), the notion of propagating fields came into clear view and gained widespread acceptance. Newtonian action-at-a-distance gravity itself then yielded, by reason of its noncovariance under special relativity, and became a propagating wave theory in the non-Euclidean forge of general relativity (1915). The original Newtonian corpuscular theory of light that fell into disfavor after the advent of Maxwellian theory was revived with the evidence of the photoelectric effect and its satisfactory interpretation by Einstein in 1905. The resulting dual view of light as both a corpuscular and continuous wave phenomenon led to the extension of these features to matter by deBroglie in 1924, with the consequence that particle-based quantum mechanics was able to emerge under the efforts of Schrödinger, Heisenberg, and others. In essence, wave-particle duality became—and still is—a universality, an article of faith of such strength that all interaction mechanisms were assumed to have an underlying *exchange particle* or *field quantum,* localized in physical extent, yet guided by or generated from fields which propagate at the speed of light. (This faith has, as is well known, even been extended to gravity with the introduction of the *graviton* exchange particle, whose existence has not yet been established experimentally.)

All of these observations tend to not only strongly disfavor the notion of instantaneous action at a distance, but virtually annihilate it, with its implicit lack of a postulated mechanism for the transmission of interaction effects. Since instantaneous action at a distance is a concept which refers to the immediate or present-time ("now") exchange of forces between objects, there can be no time delay involved, and therefore no transit time for any force delivering intermediary, such as a wave or particle. If a particle, wave, or corpuscular wave particle were transmitted instantaneously between two objects, it would appear to violate virtually all accepted laws of physics, not the least of which is the velocity bound presented by the speed of light. Thus, with the wave-particle dualism appearing so firmly entrenched both theoretically and experimentally, any would-be instantaneous action-at-a-distance theory would seem to be utterly doomed to failure at the outset.

Despite these contrary indications, this paper will show that, for certain classes of field theories, an instantaneous action-at-a-distance representation of the physical fields is not only always possible, but also is completely equivalent to the customary speed-of-light-limited retarded-time representation of fields over appropriately qualified regions of space and time. This will be achieved by making implicit the explicit delayed (retarded-time) effect; that is, by dispensing with direct analytical references to propagating effects. It will be shown, furthermore, that this can be accomplished with the very wave equations which are so highly suggestive of propagating phenomena, and done without altering their substantive content. This present paper will develop and validate the procedure for doing this and will also apply that procedure to the important subject of electromagnetism.

To make clear what is to be accomplished and to avoid confusion as to what is meant by action at a distance, consider the paradigmatic action-at-a-distance field defined by Newton's law of gravity in the form of Poisson's equation.

$$\nabla^2\phi=4\pi G\rho(t) \ , \tag{1.1}$$

whose solution is

$$\phi=-G\int\frac{\rho(t)}{r}dv \ . \tag{1.2}$$

What makes the field given by Eq. (1.2) an action-at-a-distance representation? Simply, it is the fact that ρ is an *explicit* function of the present time t; that is, upon evaluation of the integral over all space, the resulting expression is a function of the field point coordinates (via r) and, independently, the present time t. For a given time t, then, the field ϕ is known throughout all space "instantaneously." The present (time t) state of the mass distribution ρ determines ϕ everywhere "now."

What difference arises when the Laplacian operator of Eq. (1.1) is replaced with the d'Alembertian operator, so that

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=4\pi G\rho(t) \ , \tag{1.3}$$

and a wave equation is obtained? The solution

$$\phi=-G\int\frac{\rho(t')}{r}dv \tag{1.4}$$

is in a form in which ρ is an *implicit* function of the present time t via the *retarded-time* variable t', given by

$$t'=t-r/c \ . \tag{1.5}$$

To evaluate the integral, it is clear that a knowledge of ρ at the present time is not enough to determine ϕ. One must consider the source function ρ at past times t' such that the source influence at time t' *propagates* at speed c over distance r to arrive at the field point at time $t'+r/c=t$; that is, "now." This interpretation of what must be done to evaluate the integral of Eq. (1.4) is, it must be emphasized, an *interpretation,* albeit a quite natural one. Implicit in the interpretive procedure is the notion of propagating effects and the existence of underlying wavelike phenomena propagating at speed c. None of these interpretations or conclusions are to be disputed by this present paper. They are, in fact, quite "correct."

Consider, on the other hand, what it means if it were possible to write

$$\rho(t')=\rho'(t) \ , \tag{1.6}$$

that is, if from the function $\rho(t')$, computable only at retarded times t', it was possible to construct a new function $\rho'(t)$, equivalent to $\rho(t')$, but computable everywhere at present time t. For this situation Eq. (1.4) would become

$$\phi = -G \int \frac{\rho'(t)}{r} dv \tag{1.7}$$

and would fall under the paradigmatic solution offered by Eq. (1.2)—with $\rho(t)$ becoming $\rho'(t)$, thereby allowing Eq. (1.7) to also be classified as an action-at-a-distance representation of the solution to Eq. (1.3). Since the field solutions given by Eqs. (1.4) and (1.7) are the same, by reason of Eq. (1.6), it follows that either a speed-of-light-limited retarded-time interpretation or an instantaneous action-at-a-distance interpretation of the field representation may be invoked, depending only upon interpretive convenience.

With the above clarifying remarks in place, the dual interpretation of solutions to the wave equation as either propagating waves or action-at-a-distance effects is seen to be possible if a ρ' satisfying Eq. (1.6) exists. Does such a ρ' exist? For the case of Eq. (1.3) the answer is yes. The Appendix to this paper displays the structure of ρ' for two cases: one in which ρ is a continuous source function and one in which ρ represents a particulate (singular) point source. In both cases, it will be seen, infinite series are involved in the construction of both ρ' and ϕ. Since the issue of series convergence is of considerable importance in establishing a well-defined ρ' or ϕ, much of this paper develops an approach which will allow this issue to be addressed for functional expansions similar to those of ϕ, while presenting several alternative methods for generating those expansions. Throughout, the emphasis will be placed upon particulate sources, however, since this case is the more fundamental, though it is more difficult to treat.

Given the validity of the dual representation for certain fields as propagating wave or action-at-a-distance constructs, it is clear that the latter construct appears to eliminate the particle aspect of the wave-particle dualism, since instantaneous effects are naturally devoid of propagational referents or constructs—inclusive of particles or particlelike phenomenona. Furthermore, as will be seen in the body of this paper, field effects are so closely tied to their source particle kinematics that even wavelike constructs become obscured. That is, the wavelike solutions of the field equations also lose their clear identity. Therefore the concept of a field quantum itself appears vitiated. Such appearances are illusory, however, for it may be reasonably argued that *if an action-at-a-distance representation of fields is possible, the underlying field theory cannot then admit of a field quantum, and the underlying field equations cannot then be considered wholly correct.* Evidence of field quantum phenomena is so extensive that this thesis may be held demonstrably correct, until shown to be otherwise. Classical electromagnetic-field theory, for example, falls into the class of wave theories considered in this paper. An action-at-a-distance representation for its fields and potentials is possible, as will be shown. The conclusion that it admits of no field quantum (the photon) is consistent with the experimental facts, for the photoelectric and other effects cannot be explained based upon Maxwell's classical equations alone. These difficulties are well known and attempts at this resolution have guided the formulation of acceptable particle-based quantum field theories and the continuous-field-based interaction field theories of modern physics.

B. Why instantaneous action at a distance?

Since one of the hallmarks of a canonically correct field theory is the emergence of field quanta from its structure, and since the convertible (to action-at-a-distance representation) wave theories considered in this paper will not intrinsically possess this necessary structure, why then follow this course of development?

On a practical (engineering) level, the present-time representation of fields can be extremely convenient, since retarded-time representations, while formally correct, only present another problem to be solved (for the retarded time) before computational use can be made of the field expressions. Theoretically, in those cases where detailed particle structure is not a constraining issue, a present-time representation of field structures presents a uniform (present-time) basis for analytical investigations. In any circumstance where charged particles are known to preserve their structural integrity (i.e., where creation or annihilation events do not occur), an action-at-a-distance representation of particle generated fields may prove beneficial. Wave theories of matter and field quanta may benefit as well, however, from such a representation.

Wave theories whose fields obey the inhomogeneous wave equation have two common forms of source representation: macroscopic continuum sources and particle sources. The former are a convenient abstraction for reducing wave theories to practice. The latter, with which this paper shall deal, more closely underlie actual physical realities, as has been amply demonstrated in recent times. When particulate sources are invoked in any wave theory, however, two difficulties arise. First, the detailed structure of "particles" is not known, so that developing a continuum (though localized) source function for them is obstructed by lack of knowledge of their physical extent and intrinsic characteristics. The best that can be done, in most circumstances, is to adopt the abstraction of a "point" particle, with its well-known attendant field singularities. Second, even with the assumption of a point particle, solving the underlying field equations with singular point sources results in "indirect" solutions, expressed in terms of retarded (or advanced) times.

To eliminate the indirectness presented by retarded or advanced times, an instantaneous action-at-a-distance theory is required, whereby these times are uniformly and consistently evaluated "now;" that is, instantaneously in the present time of some observer. Doing so, despite the loss of field quantization capability, would allow for the development of the mechanics and dynamics of instantaneously interactive (particle-to-particle) systems. In turn, this would allow for the development of system Lagrangians and Hamiltonians which form the mainstay of much of modern particle physics—in the development of quantum field theories, for example. Instantaneous action-at-a-distance theory would also allow for a more uniform and orderly search for and examination of system invariants—without the complications attendant upon retarded- and advanced-time representations.

It is in the area of quantum field theory development that one comes full circle to the first problem mentioned above, for it has been in the quantum-mechanical representation of system Hamiltonians that both particle structures and field quanta have come to be (at least partially) understood, with respect to both extent and internal characteristics. Thus it is that instantaneous action-at-a-distance theory may come to be of some importance in extending the knowledge base. With this preface to the material, wave and instantaneous action-at-a-distance theories will now be formally described.

C. Formal developments and outline of paper

Wave theories are abstract representations of propagating wave phenomena. The representation is usually mathematical and generally takes the form of partial differential equations relating abstract fields. Abstract fields are space-time-dependent functions whose structures are so constrained by the governing differential equations as to manifest some (or perhaps all) of the salient features of the underlying wavelike phenomenon. Whether one or more field components are involved in a field description of a wave phenomenon, each component satisfies an inhomogeneous wave equation which determines the field-to-source relationship; and it is this property of the field components which will be used to characterize a wave theory for the purposes of this present paper.

If the sources are abstracted to point particles, the source term of the inhomogeneous wave equation contains a Dirac δ function as a factor, and the most general wave equation governing a source-to-field relationship is of the type given by Eq. (1.8). It is with such wave equations that this paper will exclusively deal; that is, with equations of the point source type.

The solution of the inhomogeneous wave equation,

$$\Box^2\phi=\nabla^2\phi-\frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}=-4\pi s(t)\delta(\mathbf{r}_F-\mathbf{r}_p(t)) \;, \quad (1.8)$$

in terms of retarded-time formulations is well known [3]. The field ϕ, due to a particle (p) of source strength $s(t)$, is given by

$$\phi(\mathbf{r}_F,t)=\frac{s(t')}{|\mathbf{r}_F-\mathbf{r}_p(t')|+(1/v)\dot{\mathbf{r}}_p(t')\cdot[\mathbf{r}_p(t')-\mathbf{r}_F]} \quad (1.9)$$

at the (assumed) fixed field point $\mathbf{r}_F$. t' is the retarded time, determined by solving the retarded-time equation

$$t'=t-\frac{1}{v}|\mathbf{r}_F-\mathbf{r}_p(t')| \;. \quad (1.10)$$

Given that a wave theory is restricted to a consideration of point particle sources, so that its field satisfy Eq. (1.8), it then becomes possible to convert it into an instantaneous action-at-a-distance theory. An instantaneous action-at-a-distance theory is one whose fields (the same as for the underlying wave theory) satisfy Eq. (1.8), but whose solutions are given entirely in terms of the present (source particle) time, thereby effecting an instantaneous (present-time) interaction between source and field (observation) points.

It follows that any wave theory with particle sources may be translated into a mathematically equivalent instantaneous action-at-a-distance theory by the simple mechanism of solving Eq. (1.10) for t' and then substituting the solution into Eq. (1.9). The resulting field solution will then be given strictly in terms of the present time t and the impression of instantaneity (i.e., instantaneous action at a distance) will have been effected. (It is to be noted at this point that the appellations "present time" and "instantaneous action at a distance" may be used interchangeably.)

The difficulty involved with executing the above procedure is that Eq. (1.10) cannot be solved for t' exactly, except in a relatively few cases, because of its inherently transcendental nature. Even in those relatively few cases where it can be solved exactly, the task of reducing the right member of Eq. (1.9) to a manageable form often becomes onerous. It is desirable, therefore, to present a means for not only solving Eq. (1.10) in a general way, but also for reducing the right member of Eq. (1.9) to a convenient and manageable form. Both of these tasks will be addressed in this paper, with relatively simple restrictions on the function r, defined by Eq. (1.11a) below.

Besides accomplishing the above tasks, the present paper will also apply the results to the important subject of electromagnetism whose defining fields and potentials, under a Lorentz gauge, satisfy inhomogeneous wave equations. (In electromagnetism, the retarded potential is a four-vector, generally referred to as the Liénard-Wiechert potentials.) It should be remembered, however, that although electromagnetic applications are stressed in this paper, the results will apply equally well to any field phenomenon which is governed by an inhomogeneous wave equation of the type presented in Eq. (1.8); this is, with particle sources. To emphasize this point, the retarded-time solution to Eq. (1.8) will be presented at this time, in terms of the mathematical formalism introduced in this paper. It is

$$\phi(\mathbf{r}_F,t)=D_m(s(t)r^{m-1}) \;, \quad (1.11)$$

with

$$r=|\mathbf{r}_F-\mathbf{r}_p(t)| \quad (1.11a)$$

and

$$D_m=\frac{(-1)^m}{v^m m!}\frac{d^m}{dt^m} \;, \quad (1.11b)$$

with summation on the index m from 0 to ∞ being implied in Eq. (1.11). The formal derivation of this equation is contained in the sections that follow.

When Eqs. (1.11) are examined, it is to be noted that they are expressed entirely in terms of the present time (or particle time) t. That is, they make no reference to the retarded time t'. Essentially then, the formalism to be introduced herein removes retarded-time references from field computations and expresses the fields directly in terms of the present (particle) time. Therefore this formalism will be found to constitute an adequate basis for an instantaneous action-at-a-distance theory of particle-to-particle interactions.

Only the electromagnetic fields generated from the arbitrary motion of charged particles moving *in vacuo* will be treated in this paper. Restricting particles to a vacuum is tantamount to restricting their speed to the signal propagation speed c, so that shock effects, such as Cherenkov radiation, must be developed as an application of the following material, and should not be considered an intrinsic part of it. (With no loss of generality, the assignment $v=c$, the vacuum speed of light, will be maintained throughout the rest of this paper.)

A treatment of particle dynamics is omitted from the material that follows. This omission is compatible with the field view of particle-to-particle interactions. In this view, fields (or field quanta) are seen as being generated from their particle sources, and then the fields (or field quanta) so generated are perceived as interacting with some target particle. This two-part representation of particle interactions is didactically convenient, and has been adopted in this paper to reduce the subject material into a manageable presentation. The present material, it has proved, is readily expoundable if the target particle reaction to (or interaction with) the source fields is excluded. Some of the complexities that can result when a total particle-to-particle interaction theory is undertaken are ably described by Whittaker [4] in his compendious treatment of the history of electromagnetism. Sommerfeld's electrodynamics [5] contains a more abbreviated (though more technically formal) discussion of some of those same complexities. The difficulties, of course, are not restricted to electromagnetic-field theory, but apply almost universally, given the ubiquitousness of Eq. (1.8). They will be avoided in this present treatment, however, which may then be considered an effort to exactly solve Eq. (1.8) in an instantaneous action-at-a-distance or present- (particle) time format, wherein reference to retarded time is eliminated from the field solution.

As already mentioned, the key to following this agenda rests with the solution of the classical retarded-time equation. This equation is derived in the next section, along with its heuristic solution in terms of Lagrange series. In Sec. III, a formal solution is provided, with extensions. In Sec. IV, the classical Liénard-Wiechert potentials are presented in present-time format, with all reference to retarded times removed. The electromagnetic fields are also developed in present-time format and are shown to be equivalent to the classical retarded-time representation of the fields. Section V presents some examples which show that the new formulation reduces to familiar results for the examples chosen and validates the relativistic correctness of the formulations developed for the special (but important) case of uniform source particle motion. In Sec. VI new representations of the electromagnetic fields and potentials, based upon the Lagrange series formalism, are developed. Advanced potentials are introduced and discussed in Sec. VII. Their significance to field theories is addressed and the possibility is raised of performing experiments to determine if advanced (electromagnetic) potentials have phenomenological reality. Equations necessary to undertake such experimentation are presented, but not reduced to experimental detail.

II. THE RETARDED-TIME EQUATION

Suppose a charged particle to be at position $\mathbf{r}_p(t)$ at time t. At some arbitrary but fixed field point $\mathbf{r}_F$, electromagnetic effects experienced at time t, due to the charge, were generated at some earlier time, $[t]$. These effects propagated a distance $|\mathbf{r}_F-\mathbf{r}_p([t])|$ to the field point and took a time $|\mathbf{r}_F-\mathbf{r}_p([t])|/c$ to get there. This time must be the same as $t-[t]$, the time elapsed from the retarded time to time t, at which time the fields are observed. Therefore

$$[t]=t-r([t])/c \ , \tag{2.1}$$

where

$$r([t])=|\mathbf{r}_F-\mathbf{r}_p([t])| \ . \tag{2.2}$$

Equation (2.1) is the retarded-time equation which must be solved if any headway is to be made in an instantaneous action-at-a-distance formulation of electrodynamics and other field theories. A clue as to how the above equation may be solved is obtained by assuming

$$[t]=t+\sum_{n=1}^{\infty}\frac{a_n(t)}{c^n} \ . \tag{2.3}$$

Equation (2.1), with the substitution suggested by Eq. (2.3), becomes

$$\sum_{n=1}^{\infty}\frac{a_n(t)}{c^{n-1}}=-r\left[t+\sum_{n=1}^{\infty}\frac{a_n(t)}{c^n}\right] \ . \tag{2.4}$$

A Taylor series expansion of the right member about t, in powers of the summed expression, and an equating of the coefficients of like powers of c results in

$$\begin{aligned} a_1&=-r(t) \ ,\\ a_2&=\frac{1}{2!}D[r^2(t)] \ ,\\ a_3&=-\frac{1}{3!}D^2[r^3(t)] \ ,\\ &\vdots\\ a_n&=\frac{(-1)^n}{n!}D^{n-1}[r^n(t)] \ , \end{aligned} \tag{2.5}$$

where $D^n=d^n/dt^n$. Then Eq. (2.3) becomes

$$[t]=t+\sum_{n=1}^{\infty}\frac{(-1)^n}{c^n n!}D^{n-1}[r^n(t)] \ . \tag{2.6}$$

Equation (2.6) is the correct solution to Eq. (2.1). Of course, this has not been rigorously demonstrated, only suggested. The structure of the general term written in

Eqs. (2.5) has not been proven, and if it were, proof of series convergence would still be lacking. Furthermore, Eq. (2.6) itself is not all that is desired. It will be convenient to evaluate certain functions at retarded time $[t]$, and to have present-time expressions for those evaluations. All of these aims can be met by recognizing that the sum in Eq. (2.6) has the structure of a *Lagrange series* [2,6], pertaining to which the methods of analytic function theory will be found sufficient in providing an elegant and rigorous proof of the equation, in establishing the conditions of its convergence, and in developing present-time formulation of those particular retarded-time functions needed for electromagnetic and other field-theoretic applications.

III. SOLUTION OF THE RETARDED-TIME EQUATION

The following theorem and proof will be sufficient to establish the paper's instantaneous action-at-a-distance field formulas.

A. An inversion theorem for analytic functions

Let the functions $f(z)$ and $g(z)$ be regular in a simply connected region containing a simple closed path C, and let z_0 be an interior point of the open subregion bounded by C. Then there exists a neighborhood of infinity K such that for σ belonging to K, the regular function

$$F(z)=z-z_0-f(z)/\sigma \tag{3.1a}$$

has one and only one zero, $z=z'$, in the closed region bounded by C. This zero is of multiplicity one and is given by

$$z'=z_0+\sum_{n=1}^{\infty}\frac{1}{\sigma^n n!}D^{n-1}[f(z_0)^n]\ , \tag{3.1b}$$

where $D^n=d^n/dz_0^n$. Furthermore,

$$g(z')=\sum_{n=0}^{\infty}\frac{1}{\sigma^n n!}D^n\left[g(z_0)\left[1-\frac{f'(z_0)}{\sigma}\right]f(z_0)^n\right]\ , \tag{3.1c}$$

or equivalently,

$$g(z')=g(z_0)+\sum_{n=1}^{\infty}\frac{1}{\sigma^n n!}D^{n-1}[g'(z_0)f(z_0)^n]\ , \tag{3.1d}$$

where primes on f and g indicate ordinary differentiation.

The series indicated in Eqs. (3.1b)–(3.1d) are convergent for any σ in K. The region K is determined by $|\sigma|>R$, where R is the least upper bound of the real set $|f(z)/(z-z_0)|$, mapped from points z of C.

Proof: Since $f(z)$ is regular in the closed region bounded by C, it follows [7] that $f(z)$ attains its maximum modulus M on C. Also, if z lies on C then, because z_0 is interior to C, $|z-z_0|$ attains its greatest lower bound, $\lambda\neq 0$, on C. Let $Q(z)=f(z)/(z-z_0)$. Then for all points on C, $|Q(z)|\leq M/\lambda$ and $|Q(z)|$ is bounded above.

Now let R be the least upper bound of the real number set formed from $|Q(z)|$ when z lies on C. This least upper bound is attained by at least one point on C and, therefore, $|Q(z)|\leq R$ for all z on C. Let σ be a complex number arbitrarily chosen from the neighborhood of infinity defined by $K=\{\sigma\,|\,|\sigma|>R\}$. It follows that

$$\left|\frac{f(z)}{\sigma(z-z_0)}\right|<1 \tag{3.2}$$

for all points on C.

By Rouché's theorem [8], the satisfaction of inequality (3.2) guarantees that the function defined by Eq. (3.1a) possesses one and only one zero, z', of multiplicity one, within the region bounded by C. Furthermore, there are no zeros of $F(z)$ on C, or else condition (3.2) is violated.

Because of the regularity of $F(z)$ at z', the zero there may be factored in the form

$$F(z)=(z-z')G(z)\ , \tag{3.3}$$

where $G(z)$ has neither zeros nor poles in the closed region bounded by C. Because $G(z)$ has no zeros or poles in this region, the function $G'(z)/G(z)$ is regular there. That fact may then be utilized to establish the series expansions represented by Eqs. (3.1b)–(3.1d).

Using Cauchy's integral formula and theorem there follows

$$\begin{aligned}
g(z')&=\frac{1}{2\pi i}\int_c\frac{g(z)}{z-z'}dz\\
&=\frac{1}{2\pi i}\int_c g(z)\left[\frac{1}{z-z'}+\frac{G'(z)}{G(z)}\right]dz\\
&=\frac{1}{2\pi i}\int_c g(z)\frac{F'(z)}{F(z)}dz\quad[\text{by (3.3)}]\\
&=\frac{1}{2\pi i}\int_c g(z)\frac{1-f'(z)/\sigma}{z-z_0-f(z)/\sigma}dz\quad[\text{by (3.1a)}]\\
&=\frac{1}{2\pi i}\int_c\frac{g(z)}{z-z_0}\frac{1-f'(z)/\sigma}{1-f(z)/\sigma(z-z_0)}dz\\
&=\frac{1}{2\pi i}\int_c\frac{g(z)}{z-z_0}[1-f'(z)/\sigma]\sum_{n=0}^{\infty}\left[\frac{f(z)}{\sigma(z-z_0)}\right]^n dz\\
&\qquad[\text{by (3.2)}]\ .
\end{aligned}$$

The integrations may be carried out term by term because of the uniform convergence of the series for all points z on C. There then follows:

$$\begin{aligned}
g(z')&=\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_c\frac{g(z)[1-f'(z)/\sigma][f(z)/\sigma]^n}{(z-z_0)^{n+1}}dz\\
&=\sum_{n=0}^{\infty}\frac{1}{\sigma^n n!}D^n\left[g(z_0)\left[1-\frac{f'(z_0)}{\sigma}\right]f(z_0)^n\right]\ ,
\end{aligned}$$

which is result (3.1c). Expanding terms of the above expression, there also follows:

$$g(z')=\sum_{n=0}^{\infty}\frac{1}{\sigma^n n!}D^n[g(z_0)f(z_0)^n]$$
$$-\sum_{n=0}^{\infty}\frac{1}{\sigma^{n+1}(n+1)!}D^{n+1}[g(z_0)f(z_0)^{n+1}]$$
$$+\sum_{n=0}^{\infty}\frac{1}{\sigma^{n+1}(n+1)!}D^n[g'(z_0)f(z_0)^{n+1}]$$
$$=g(z_0)+\sum_{n=1}^{\infty}\frac{1}{\sigma^n n!}D^{n-1}[g'(z_0)f(z_0)^n] ,$$

which is result (3.1d). Letting $g(z)=z$ in Eq. (3.1d), result (3.1b) follows, and the theorem is proved.

It should be noted that virtually the same results may be obtained by the much simpler and more direct method of taking the Fourier transform of the retarded function $g(t')$, where $t'=[t]$. In fact, with $\mathcal{F}$ designating the Fourier transform and with Eq. (2.1) written in the form

$$t=t'+r(t')/c ,$$

one readily obtains

$$\mathcal{F}g(t')=\int_{-\infty}^{\infty}g(t')e^{-i\omega t}dt$$
$$=\int_{-\infty}^{\infty}g(t')\left[1+\frac{\dot r(t')}{c}\right]e^{-i\omega[t'+r(t')/c]}dt'$$
$$=\int_{-\infty}^{\infty}g(t)\left[1+\frac{\dot r(t)}{c}\right]e^{-i\omega[t+r(t)/c]}dt$$
$$=\int_{-\infty}^{\infty}g(t)\left[1+\frac{\dot r(t)}{c}\right]\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}(i\omega)^n r^n(t)\right]e^{-i\omega t}dt$$
$$=\mathcal{F}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}(i\omega)^n\left[g(t)\left[1+\frac{\dot r(t)}{c}\right]r^n(t)\right]\right]$$
$$=\mathcal{F}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}D^n\left[g(t)\left[1+\frac{\dot r(t)}{c}\right]r^n(t)\right]\right] ,$$

so that

$$g([t])=\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}D^n\{g(t)[1+\dot r(t)/c]r^n(t)\} ,$$

in agreement with Eq. (3.1c). However, Eq. (3.1c) is more general, in that it admits complex time values and so allows mathematical investigations to proceed into the complex plane, where series convergence criteria may be readily established, as will be shown in Sec. III C below, and where multiple roots of the retarded-time equation may be treated, as in Sec. III D.

B. Retarded-time series solution

To make use of the inversion theorem, first observe that solving Eq. (2.1) is equivalent to finding a zero, $z'=[t]$, of the function

$$F(z)=z-t+r(z)/c . \quad (3.4)$$

This will be in the form of Eq. (3.1a) with the associations

$$z_0=t, \quad f=r, \quad \text{and } \sigma=-c . \quad (3.5)$$

The inversion theorem may then be applied if one assumes that the function $r(z)$ is regular in a region containing the real (time) axis or an appropriate portion thereof.

Making the above associations and assumption, Eq. (3.1b) translates precisely into Eq. (2.6) of the preceding section. It is noteworthy that the inversion theorem guarantees that the series of Eq. (2.6) always converges if c is "large enough"; that is, if it is an element of the set K, previously defined. Therefore, as a practical matter, one may always assume beforehand that c is "large enough" and that Eq. (2.6) is therefore valid. If any divergence of the series occurs as c is reduced to its correct speed-of-light value, then one may minimally conclude that c was not "large enough" to present the problem in convergent series form. Nevertheless, it may sometimes be possible to cast a presenting series with a c not "large enough" into another form—effectively, an analytic continuation—for which the correct speed-of-light value for c is adequate in establishing a valid convergent series or closed form solution. An example of such analytic continuation to a closed form will be presented in Sec. V.

In the event that analytic continuation of a Lagrange series to another convergent series or to a closed form is not feasible, detailed testing of the validity of inequality (3.2) may be performed, with due cognizance taken, in the construction of an integration contour, of all branch points and poles of the function $r(z)$. Such testing is appropriate if convergence of the Lagrange series is to be assured when c realizes its correct speed-of-light value. It is also appropriate in the estimation of errors when truncating the series. A brief example of such contour testing will be presented at the end of Sec. V. Other than this, examples of contour testing will not be included in this paper, since the subject of series convergence by means of contour testing can become rather extensive in its own right.

As an alternative to contour testing, the Lagrange series may be directly tested for convergence as given, whenever this proves more convenient. Standard series convergence tests, such as the ratio or root test—or series comparison tests—may then be applied, whenever feasible.

C. Strength of Lagrange series convergence

In deriving the retarded-time expansions in Sec. III A, use was made of a contour integral which, in general form, may be expressed

$$\frac{1}{2\pi i}\int_c \frac{\phi(z)}{(z-z_0)^{n+p+1}}\left[\frac{f(z)}{\sigma}\right]^n dz = \frac{1}{\sigma^n(n+p)!}D^{n+p}[\phi(z_0)f(z_0)^n] , \quad (3.6)$$

where $D=d/dz_0$, $\phi(z)$ is some regular function on and within the contour C, and the equality results from applying Cauchy's integral formula. It has already been stated in Sec. III A that the regularity of the function $f(z)$ on and within C implies a bound, M, of its modulus on C, and that $|z-z_0|$ is bounded below by λ. Since $\phi(z)$ is assumed regular in the same region, it likewise has a modular bound, B, on C. Assuming that contour C is rectifiable and of length Γ, it follows [7] that the modulus of the left number of Eq. (3.6) is bounded above by $\Gamma B M^n/2\pi|\sigma|^n\lambda^{n+p+1}$. The right number of Eq. (3.6) is then also so bounded, so that

$$\left|\frac{1}{\sigma^n(n+p)!}D^{n+p}[\phi(z_0)f(z_0)^n]\right| \le \frac{\Gamma B}{2\pi\lambda^{p+1}}\left[\frac{M}{|\sigma|\lambda}\right]^n . \quad (3.7)$$

Equation (3.7) can be useful in estimating series truncation errors, since from it there also follows

$$\left|\sum_{n=m}^{\infty}\frac{1}{\sigma^n(n+p)!}D^{n+p}[\phi(z_0)f(z_0)^n]\right| \le \frac{\Gamma B|\sigma|}{2\pi\lambda^p(|\sigma|\lambda-M)}\left[\frac{M}{|\sigma|\lambda}\right]^m , \quad (3.8)$$

which bounds the error due to truncation of the series after the mth term (assuming series summation from 0 to ∞), provided $M/|\sigma|\lambda<1$. From Sec. III A, however, with σ selected from the set K,

$$\left|\frac{f(z)}{\sigma(z-z_0)}\right| \le \frac{M}{|\sigma|\lambda}<1 \quad (\text{for } z \text{ on } C) . \quad (3.9)$$

Therefore relation (3.9) [or relation (3.2)] not only guarantees the existence of a retarded-time solution of root multiplicity one and the convergence of all retarded-time functional expansions, but it also determines the strength of series convergence through relation (3.8) and bounds moduli of terms of the series through relation (3.7).

D. Multiple roots

In the event that the retarded-time equation possesses multiple roots and it is known, for a given σ, that the associated Lagrange series converges, then to which of the multiple roots does it converge? The question implicitly assumes a lack of knowledge of the contours which guarantee convergence, for if any were known, the applicable root would be constrained to lie within its boundaries and thereby be isolated and identified.

The above question will be answered for the special case (relevant to physical applications) in which the associations given by Eqs. (3.5) apply. Since r is assumed analytic and $r(t)$ represents spatial separation at (real) time t, it follows from the reflection principle of analytic function theory [9] that, with $*$ indicating complex conjugation,

$$[r(z)]^*=r(z^*) . \quad (3.10)$$

Then,

$$\left|\frac{r(z^*)}{c(z^*-t)}\right| = \left|\frac{[r(z)]^*}{[c(z-t)]^*}\right| = \left|\frac{r(z)}{c(z-t)}\right| , \quad (3.11)$$

so that, if the convergence requirement [Eq. (3.2) or (3.9)] is satisfied for any z, it is also satisfied for z^*, the conjugated complex time point.

Since t is real, any simple contour which surrounds point $(t,0)$ must intersect the real axis at least once on either side of this point. The discussion will be restricted to those contours which intersect the real axis precisely once on either side of $(t,0)$. Let it be assumed that one such contour guarantees convergence of the Lagrange series. Regardless of its initial shape, it may be replaced by a contour which is symmetric about the real time axis by reason of Eq. (3.11) (by mirror imaging either the upper or lower half). By construction, the resulting contour also guarantees convergence of the Lagrange series. In addition, the interior of the symmetric contour fully contains the real axis between the contour intersection points with the real axis.

The interior of the symmetric contour can contain no complex roots, for when t is real, the retarded-time equation

$$z=t-r(z)/c \quad (3.12)$$

is also satisfied by z^* [by Eq. (3.10)] and the interior cannot contain a double root, by reason of Rouché's theorem (within the context of the inversion theorem). Therefore the interior of the contour must contain precisely one real root, less than t [by reason of Eq. (3.12)], and this must be the real root which lies closest to $(t,0)$, for if it were not, then at least two (retarded) roots would again be contained within the contour, in violation of the inversion (Rouché) theorem.

Therefore, under the conditions imposed above (which should apply to all cases of physical interest), *the Lagrange series expansion for the retarded time will always converge to that real root which is less than and nearest to the present time t, regardless of the number of distinct roots which satisfy the retarded-time equation.*

As an example, consider a uniformly accelerated particle whose coordinates are given by $(\frac{1}{2}at^2,0,0)$. If the origin is the field (observation) point, then $r(t)=\frac{1}{2}at^2$ and Eq. (3.12) may be readily solved to yield the retarded times

$$t_1=\tau(\sqrt{1+2t/\tau}-1) \tag{3.13a}$$

and

$$t_2=-\tau(\sqrt{1+2t/\tau}+1)\ , \tag{3.13b}$$

where $\tau=c/a$.

According to the preceding proof, the Lagrange series expansion for the retarded time, Eq. (2.6), must result in

$$[t]=t_1\ . \tag{3.14}$$

To show this, substitute $r(t)$ into Eq. (2.6) so that

$$\begin{aligned}[t]&=t+\sum_{n=1}^{\infty}\frac{(-1)^n}{c^n n!}D^{n-1}\left[\frac{1}{2}at^2\right]^n\\ &=t+\sum_{n=1}^{\infty}\frac{1}{n!}\left[-\frac{1}{2\tau}\right]^n D^{n-1}t^{2n}\\ &=t-2\tau\sum_{n=1}^{\infty}\frac{(2n)!}{n!(n+1)!}\left[-\frac{t}{2\tau}\right]^{n+1}\ .\end{aligned} \tag{3.15}$$

The ratio test shows the above series to be convergent for $|t|<|\tau|/2$.

For $|x|<1$, the following binomial expansion is easily verified:

$$\sqrt{1+x}=1+\frac{1}{2}x+\sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{2^{2n+1}n!(n+1)!}x^{n+1}\ . \tag{3.16}$$

Comparing Eqs. (3.15) and (3.16) (with $x=2t/\tau$), Eq. (3.14) results, as was to be shown.

For the example chosen, if $a>0$, the particle decelerates to the spatial origin during negative times and accelerates from the origin during positive times, arriving at the origin at $t=0$. The retarded times given by t_1 in Eq. (3.13a) are physically meaningful so long as $t\geq-\tau/2$. The associated Lagrange series, however, does not converge for large positive times. The right member of Eq. (3.13a) therefore represents the appropriate analytic continuation of the associated Lagrange series for times $t>\tau/2$. Notice that if c were adjustable, so that as $c\to\infty$, $\tau\to\infty$, then the series could be made to converge for any time t in accordance with the definition of the region K, defined by the inversion theorem.

The root t_2, not selected by the convergent Lagrange series, is nonphysical, since the velocity of the particle at time t_2 is given by

$$v=at_2=-c(\sqrt{1+2t/\tau}+1)\leq -c\ . \tag{3.17}$$

If the source particle could travel faster than the speed of the signals it emits, root t_2 might then be meaningful. However, the present paper deals exclusively with charged particles *in vacuo,* so this root must be excluded from physical consideration.

IV. ACTION-AT-A-DISTANCE INSTANTANEOUS POTENTIALS AND FIELDS

The classical Liénard-Wiechert potentials, expressed in mks units, are given by

$$V(t)=\frac{q}{4\pi\epsilon_0}\frac{1}{[r](1+[\dot r]/c)} \tag{4.1}$$

and

$$\mathbf{A}(t)=\frac{\mu_0 q}{4\pi}\frac{[\mathbf{v}_p]}{[r](1+[\dot r]/c)}\ , \tag{4.2}$$

where

$$r=r(t)=|\mathbf{r}_F-\mathbf{r}_p(t)|,\quad [r]=r([t])\ ,$$

$$[\dot r]=\dot r([t]),\quad \mathbf{v}_p=\dot{\mathbf{r}}_p\ ,$$

and

$$[\mathbf{v}_p]=\mathbf{v}_p([t])\ .$$

Using Eq. (3.1c) with $z'=[t]$, $g(z')=q/4\pi\epsilon_0[r](1+[\dot r]/c)$, and the association given by relations (3.5), there results

$$V(t)=\frac{q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}D^n(r^{n-1}) \tag{4.3}$$

and, similarly,

$$\mathbf{A}(t)=\frac{\mu_0 q}{4\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}D^n(\mathbf{v}_p r^{n-1})\ , \tag{4.4}$$

where $D^n=d^n/dt^n$. Equations (4.3) and (4.4) are the Lagrange series representations of the electromagnetic scalar and vector potentials—taken under a Lorentz gauge—which this paper wished to establish for the electromagnetic application of its material. [The Lorentz gauge condition is implicit in the expressions for the potentials given by Eqs. (4.1) and (4.2).] The series are always convergent if c is "large enough," as discussed earlier.

While Eq. (4.4), as it stands, has an electromagnetic interpretation, it is not restricted to electromagnetic applications, since it is, in fact, a solution of Eq. (1.8) with the substitutions $s\to(\mu_0 q/4\pi)\mathbf{v}_p(t)$ and $\phi\to\mathbf{A}$. Since $\mathbf{v}_p(t)$ can be an arbitrary function, Eq. (1.11) is therefore justified and validated. In addition, the functions ϕ and s need not be scalars, as is clearly demonstrated by the preceding substitutions.

The requirement for regularity in the functions defined by the right members of Eqs. (4.1) and (4.2) is met by excluding all physical situations in which $r=0$ or $\dot r=-c$. Since these singular cases are of little physical significance, the functions defined by Eqs. (4.1) and (4.2) may be considered analytically continuable from the real time axis in all circumstances; and, therefore, series (4.3) and(4.4) may be considered valid representations of the electromagnetic potentials in all circumstances in which the presenting series are convergent.

The electric and magnetic fields are given by

$$\mathbf{E}=-\nabla_F V-\frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B}=\nabla_F\times\mathbf{A} . \tag{4.5}$$

All derivatives in Eqs. (4.5) commute with D^n, because spatial differentiation is to be carried out with respect to the coordinates of $\mathbf{r}_F$ only, as is indicated by the operator subscripting. Carrying out the operations indicated above, there results

$$\mathbf{E}(t)=-\frac{q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}D^n \times\left[(n-1)r^{n-3}\mathbf{r}+\frac{1}{c^2}D(r^{n-1}\mathbf{v}_p)\right] \tag{4.6}$$

and

$$\mathbf{B}(t)=\frac{\mu_0 q}{4\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(n-1)}{c^n n!}D^n(\mathbf{r}\times\mathbf{v}_p r^{n-3}) , \tag{4.7}$$

with $\mathbf{r}=\mathbf{r}_F-\mathbf{r}_p(t)$. The structures of Eqs. (4.6) and (4.7) are direct translations of the operations indicated in Eqs. (4.5).

That these equations are equivalent to the classical retarded-time formulations for the fields will now be shown, beginning with the presentation of a general procedure for translating any classical retarded-time expression into a present-time formulation.

Using the customary bracket notation—but with a prime on the bracket—to indicate retarded-time evaluation, if some function $\mathbf{G}$ is to be evaluated at a retarded time, then Eq. (3.1c), with $\sigma=-c$ and $f=r$, may be expressed as

$$[\mathbf{G}]'=\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}D^n(\mathbf{G}\chi r^n) , \tag{4.8}$$

where

$$\chi=1+\dot{r}(t)/c , \tag{4.9}$$

or, using Eq. (3.1d),

$$[\mathbf{G}]'=\mathbf{G}(t)+\sum_{n=1}^{\infty}\frac{(-1)^n}{c^n n!}D^{n-1}(\dot{\mathbf{G}}r^n) . \tag{4.10}$$

Equations (4.8) and (4.10), with the definition provided by (4.9), comprise the means for generating present-time expressions (substituting left to right), or generating retarded-time expressions (proceeding right to left).

Because of the repetition of its occurrence, it will be convenient to introduce the differential operator,

$$D_n=\frac{(-1)^n}{c^n n!}D^n . \tag{4.11}$$

It will also be found convenient to adopt the convention of dropping the summation sign when summation on an index proceeds from 0 to ∞, provided the index is repeated within the summand. If summation proceeds from i to infinity, when $i\neq 0$, then the index i will be additionally subscripted to the lower right of the symbol D_n, which then becomes $D_{n,i}$. Then Eqs. (4.8) and (4.10) may be written, respectively,

$$[\mathbf{G}]'=D_n(\mathbf{G}\chi r^n) \tag{4.12}$$

and

$$[\mathbf{G}]'=\mathbf{G}(t)-\frac{1}{cn}D_{n-1,1}(\dot{\mathbf{G}}r^n) . \tag{4.13}$$

Using the above notation, it may be shown that, for any function $\mathbf{g}$,

$$D_n(n\mathbf{g}r^{n-1})=-\frac{1}{c}D_n\left[D\left[\frac{\mathbf{g}}{\chi}\right]r^n\right] \tag{4.14}$$

(see the Appendix for proof). A symbol D without subscript or superscript will always stand for the basic operator d/dt. For low-order time derivatives, the customary dot notation will also be used, as in Eq. (4.13).

The following relation also holds:

$$D[\mathbf{G}]'=\left[\frac{1}{\chi}\dot{\mathbf{G}}\right]' , \tag{4.15}$$

since

$$D=\frac{d}{dt}=\frac{d[t]'}{dt}\frac{d}{d[t]'}=\frac{1}{[\chi]'}\frac{d}{d[t]'}=\left[\frac{1}{\chi}D\right]' , \tag{4.16}$$

where, in computing the time derivative, use has been made of the retarded-time equation,

$$[t]'=t-\frac{1}{c}r([t]') . \tag{4.17}$$

Equation (4.6) may now be written

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}D_n\left[n\left[-\frac{\hat{\mathbf{n}}}{r}\right]r^{n-1}+\frac{\hat{\mathbf{n}}}{r^2}r^n+\frac{1}{c}D\left[-\frac{\boldsymbol{\beta}}{r}r^n\right]\right] , \tag{4.18}$$

with

$$\boldsymbol{\beta}=\frac{\mathbf{v}_p}{c} \quad \text{and} \quad \hat{\mathbf{n}}=\frac{\mathbf{r}}{r} . \tag{4.19}$$

But,

$$D_n\left[n\left[-\frac{\hat{\mathbf{n}}}{r}\right]r^{n-1}\right]=D_n\left[\frac{1}{c}D\left[\frac{\hat{\mathbf{n}}}{\chi r}\right]r^n\right] =\left[\frac{1}{c\chi}D\left[\frac{\hat{\mathbf{n}}}{\chi r}\right]\right]' , \tag{4.20}$$

by applying Eqs. (4.14) and (4.12), in succession.

Also,

$$D_n\left[\left[\frac{\hat{\mathbf{n}}}{r^2}\right]r^n\right]=\left[\frac{\hat{\mathbf{n}}}{\chi r^2}\right]' \tag{4.21}$$

by Eq. (4.12). Finally,

$$D_n\left[\frac{1}{c}D\left[-\frac{\boldsymbol{\beta}}{r}r^n\right]\right]=D\left[\frac{1}{c}D_n\left[-\frac{\boldsymbol{\beta}}{r}r^n\right]\right]$$
$$=D\left[\frac{1}{c}\left[-\frac{\boldsymbol{\beta}}{\chi r}\right]\right]'$$
$$=\left[\frac{1}{c\chi}D\left[-\frac{\boldsymbol{\beta}}{\chi r}\right]\right]' , \quad (4.22)$$

by commutativity of D and D_n, and applying Eqs. (4.12) and (4.15) in succession.

Combining results (4.20)–(4.22) into Eq. (4.18), there follows

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{n}}}{\chi r^2}+\frac{1}{c\chi}D\left[\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\chi r}\right]\right]' . \quad (4.23)$$

Equation (4.23) is, of course, one of the classical formulations for the electric field, expressed in retarded-time format [10]. Proceeding from Eq. (4.23) to Eq. (4.6) simply involves reversing the above steps.

In a similar way, Eq. (4.7) may be written

$$c\mathbf{B}=\frac{q}{4\pi\epsilon_0}D_n\left[n\left[\frac{\hat{\mathbf{n}}\times\boldsymbol{\beta}}{r}\right]r^{n-1}+\frac{\boldsymbol{\beta}\times\hat{\mathbf{n}}}{r^2}r^n\right] . \quad (4.24)$$

But

$$D_n\left[n\left[\frac{\hat{\mathbf{n}}\times\boldsymbol{\beta}}{r}\right]r^{n-1}\right]=D_n\left[\frac{1}{c}D\frac{\boldsymbol{\beta}\times\hat{\mathbf{n}}}{\chi r}r^n\right]$$
$$=\left[\frac{1}{c\chi}D\frac{\boldsymbol{\beta}\times\hat{\mathbf{n}}}{\chi r}\right]' ,$$

using Eqs. (4.14) and (4.12), in succession. Also,

$$D_n\left[\frac{\boldsymbol{\beta}\times\hat{\mathbf{n}}}{r^2}r^n\right]=\left[\frac{\boldsymbol{\beta}\times\hat{\mathbf{n}}}{\chi r^2}\right]' , \quad (4.25)$$

using Eq. (4.12). Therefore Eq. (4.24) becomes

$$c\mathbf{B}=\frac{q}{4\pi\epsilon_0}\left[\frac{\boldsymbol{\beta}\times\hat{\mathbf{n}}}{\chi r^2}+\frac{1}{c\chi}D\left[\frac{\boldsymbol{\beta}\times\hat{\mathbf{n}}}{\chi r}\right]\right]' , \quad (4.26)$$

which is one of the classical formulations for the magnetic induction, expressed in retarded-time format [10].

It has thus been shown that Eqs. (4.6) and (4.7) are completely equivalent to the classical retarded-time expressions for the electromagnetic field. They are, however, expressed in present-time format. These equations represent, therefore, one form of the instantaneous action-at-a-distance solutions of Maxwell's field equations for a charged particle in motion. All of the information contained in Maxwell's field equations are also contained in Eqs. (4.6) and (4.7), but the latter equations conveniently and completely solve Maxwell's equations, without approximation, for every definable kinematic circumstance—the requirement being, of course, that the charged particle source kinematics be known. Determining those kinematics remains, however, a central problem of electrodynamic theory, just as it is when Maxwell's field equations are used as a basis for investigations.

Conversion between retarded-time expressions and Lagrange series representations is now quite simple using Eqs. (4.12)–(4.16). Well-established results, such as

$$c\mathbf{B}=[\hat{\mathbf{n}}]'\times\mathbf{E} , \quad (4.27)$$

still hold in the new formulation and are rather simple to demonstrate. The new formulation lends itself quite well, however, to examining old formulations in new and simple ways. Examples will be presented in Sec. VI.

No mention has yet been made of the relativistic correctness of the field expressions given by Eqs. (4.6) and (4.7) or of the potentials given by Eqs. (4.3) and (4.4). The assumed covariance of Maxwell's equations assures this correctness. However, it will be beneficial to present an explicit validation of this claim, for in so doing, an example may also be provided of the benefits achievable by analytically continuing the Lagrange series solutions into closed forms, for which the requirement that c must be "large enough" may be removed. In the next section this, along with some other simple validating examples, are presented.

V. VALIDATING EXAMPLES

The issue of validity is not genuine, given the correctness of the mathematical procedures used to obtain the Lagrange series representations of the potentials. Nevertheless, it is worthwhile to instill a sense of trust in the use of new mechanisms by showing that more familiar results are obtainable from their use. Several familiar results will be offered.

Consider the specific representation of the series given by Eqs. (4.3) and (4.4):

$$V=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}+\frac{1}{2c^2}\ddot{r}-\frac{1}{3c^3}[\dot{r}\ddot{r}+3\dot{r}\ddot{r}]+\cdots\right] \quad (5.1)$$

and

$$\mathbf{A}=\frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{v}_p}{r}-\frac{1}{c}\dot{\mathbf{v}}_p+\frac{1}{2c^2}[\ddot{\mathbf{v}}_p r+2\dot{\mathbf{v}}_p\dot{r}+\mathbf{v}_p\ddot{r}]-\cdots\right] . \quad (5.2)$$

Notice that V and $\mathbf{A}$ correctly reduce to the Liénard-Wiechert potentials without time retardation as $c\rightarrow\infty$. Also, note that the classical static potentials, $V=q/4\pi\epsilon_0 r$ and $\mathbf{A}=0$, are obtained when all time derivatives of position vanish. Finally, observe that if $r=$const, $V=q/4\pi\epsilon_0 r$, as in the static case and, from Eq. (4.4),

$$\mathbf{A}=\frac{\mu_0 q}{4\pi r}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\frac{r}{c}\right]^n D^n\mathbf{v}_p$$
$$=\frac{\mu_0 q}{4\pi r}\mathbf{v}_p\left[t-\frac{r}{c}\right] .$$

Again, the resulting potentials are consistent with the Liénard-Wiechert potentials for this special case.

The next example will compare the potentials obtained from applying the principles of special relativity to those obtained from the Lagrange series representations. Evaluation of the Lagrange series expansion will additionally provide an example of Lagrange series evaluation techniques and will demonstrate analytic continuation to a closed form representation. The case chosen will be that of a charged particle moving with constant speed, v, along the positive x axis of an inertial coordinate system.

Assume all clocks to be synchronized to $t=t_p=0$ at $x=x_p=0$. The subscripted variables refer to the "moving" particle system. Assume the field point to be located in the $x=0$ plane of the "rest" system at a distance ρ from the x axis. If the positive x axes of both systems are superposed and like aligned, then the moving system will assign orthogonal coordinates, $-vt_p$ and ρ, to the position of the field point. Since the charged particle is at rest in the moving frame, the moving frame assigns a distance $\sqrt{\rho^2+v^2t_p^2}$ from the particle to the field point. Then, in its own perceived rest frame, the moving system assigns the following potentials at the field point:

$$V_p=\frac{q}{4\pi\epsilon_0\sqrt{\rho^2+v^2t_p^2}} \quad \text{and} \quad \mathbf{A}_p=\mathbf{0} \ . \tag{5.3}$$

A Lorentz transformation of the four-vector in Eqs. (5.3) using the customary assignments,

$$\beta=v/c \quad \text{and} \quad \gamma=1/\sqrt{1-\beta^2} \ , \tag{5.4}$$

results in

$$V=\gamma V_p \quad \text{and} \quad \mathbf{A}=\gamma\frac{v}{c^2}V_p(1,0,0) \ . \tag{5.5}$$

Substituting the Lorentz time transformation,

$$t_p=\gamma t \ , \tag{5.6}$$

into Eq. (5.5), there follows

$$V=\frac{q}{4\pi\epsilon_0\sqrt{\rho^2(1-\beta^2)+v^2t^2}} \tag{5.7}$$

and

$$\mathbf{A}=\frac{v}{c^2}V(1,0,0) \ . \tag{5.8}$$

In contrast to the above procedure, the Lagrange series expansion, Eq. (4.3), may be evaluated in the inertial rest frame by simply substituting the relation $r(t)=\sqrt{\rho^2+v^2t^2}$ and reducing the resulting series expression. [Note that the requirement for functional regularity in $r(t)$ is satisfied since it is always possible to contain the real time axis in an open set not containing the imaginary branch points $t_b=\pm i\rho/v$.] The procedure is not trivial, however, and it depends upon the following two relationships:

$$\frac{d^{2N}}{dx^{2N}}(1+x^2)^{N-1/2}=\frac{[(2N-1)!!]^2}{(1+x^2)^{N+1/2}} \tag{5.9}$$

and

$$\frac{1}{\sqrt{1-4x}}=\sum_{N=0}^{\infty}\frac{(2N)!}{(N!)^2}x^N \ . \tag{5.10}$$

The first is provable by induction and the second may be verified by performing a binomial expansion in the neighborhood of $x=0$. Equation (4.3) then becomes

$$\begin{aligned} V&=\frac{q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}\frac{d^n}{dt^n}(\rho^2+v^2t^2)^{(n-1)/2} \\ &=\frac{q}{4\pi\epsilon_0\rho}\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{\rho}{c}\right]^n\frac{d^n}{dt^n}\left[1+\left[\frac{vt}{\rho}\right]^2\right]^{(n-1)/2} \\ &=\frac{q}{4\pi\epsilon_0\rho}\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{v}{c}\right]^n\frac{d^n}{dx^n}(1+x^2)^{(n-1)/2} \ , \end{aligned} \tag{5.11}$$

with $x=v\,t/\rho$. If n is odd, say $n=2N+1$, then

$$\frac{d^n}{dx^n}(1+x^2)^{(n-1)/2}=\frac{d^{2N+1}}{dx^{2N+1}}(1+x^2)^N=0 \ , \tag{5.12}$$

since the highest power occurring in the polynomial expression in x is $2N$, and the $(2N+1)$th derivative will cause this, and hence all other terms, to vanish. Therefore let $n=2N$ in Eq. (5.11) so that, with $\beta=v/c$,

$$\begin{aligned} V&=\frac{q}{4\pi\epsilon_0\rho}\sum_{N=0}^{\infty}\frac{\beta^{2N}}{(2N)!}\frac{d^{2N}}{dx^{2N}}(1+x^2)^{N-1/2} \\ &=\frac{q}{4\pi\epsilon_0\rho}\sum_{N=0}^{\infty}\frac{\beta^{2N}}{(2N)!}\frac{[(2N-1)!!]^2}{(1+x^2)^{N+1/2}} \\ &=\frac{q}{4\pi\epsilon_0\rho}\frac{1}{\sqrt{1+x^2}}\sum_{N=0}^{\infty}\frac{[(2N-1)!!]^2}{(2N)!}\left[\frac{\beta^2}{1+x^2}\right]^N \ , \end{aligned} \tag{5.13}$$

using Eq. (5.9). But it may be verified that

$$\frac{[(2N-1)!!]^2}{(2N)!}=\frac{1}{4^N}\frac{(2N)!}{(N!)^2} \ , \tag{5.14}$$

so that Eq. (5.13) becomes

$$\begin{aligned} V&=\frac{q}{4\pi\epsilon_0\rho}\frac{1}{\sqrt{1+x^2}}\sum_{N=0}^{\infty}\frac{(2N)!}{(N!)^2}\left[\frac{\beta^2}{4(1+x^2)}\right]^N \\ &=\frac{q}{4\pi\epsilon_0\rho}\frac{1}{\sqrt{1+x^2}}\frac{1}{\sqrt{1-\beta^2/(1+x^2)}} \\ &=\frac{q}{4\pi\epsilon_0\rho}\frac{1}{\sqrt{1-\beta^2+x^2}}=\frac{q}{4\pi\epsilon_0\sqrt{\rho^2(1-\beta^2)+v^2t^2}} \ , \end{aligned} \tag{5.15}$$

using Eq. (5.10) and the definition of x.

Since $\mathbf{v}_p$ is a constant vector of magnitude v in the x direction, it factors out of Eq. (4.4), leaving the same series as for V except for the multiplier $\mu_0\epsilon_0\mathbf{v}_p=(v/c^2,0,0)$. Hence,

$$\mathbf{A}=\frac{v}{c^2}V(1,0,0) \ . \tag{5.16}$$

The results arrived at in Eqs. (5.15) and (5.16), using the Lagrange series expansions for V and $\mathbf{A}$, agree precisely with the results given by Eqs. (5.7) and (5.8), obtained from relativistic arguments. The Lagrange series expansions for the potentials are thus validated up to the level of application that has so far been considered. In addition, they yield, vis-á-vis special relativity, relativistically correct results. The relativistic correctness of the Lagrange series expansions for the fields and potentials is assured, in any case, since, as was already mentioned, they are exact solutions of Maxwell's equations, which themselves are taken to be covariant under the principle of relativity.

Since the steps leading to Eqs. (5.15) and (5.16) analytically continued the Lagrange series for the potentials into closed forms which remained valid for $\sigma=-c$, it was not necessary to demonstrate the existence of a contour for which Eq. (3.2) held when $\sigma=-c$. In the event that analytic continuation were not to be performed, the existence of a contour for which Eq. (3.2) holds when $\sigma=-c$ would have to be demonstrated, or independent series convergence tests would have to be applied, if series convergence is to be assured.

That there exists at least one contour for which the series converges is shown by constructing a circle about the time origin with two branch cuts, one from each branch point to the circle circumference, along the respective imaginary time axes. In this case, it is straightforward to show that as the circle radius becomes infinite, the left member of relation (3.2) is bounded above by $\beta=v/c$ over the entire contour. Thus β is at least one measure of the strength of convergence of the series, and when $\beta<1$, the series is guaranteed to converge.

VI. ALTERNATIVE FIELD FORMULATIONS IN CLASSICAL ELECTROMAGNETICS

Whether the material developed thus far has any distinct advantage over existing formulations remains to be seen. Whether new applications can be found for this present approach to electromagnetics will depend to a large extent upon the form into which it can be cast. This section presents a set of formulations for the potentials and fields which, it is hoped, will be found useful in exploring this subject further. As might be guessed, more material is omitted than is presented, but it is hoped that what is presented will provide an adequate stimulus to further investigation.

A. The potentials

The convention will be retained that any repeated index is to imply, unless otherwise indicated, summation on that index from 0 to ∞. Also, if in a defining relationship, the index occurs in a symbol being defined, then no summation of the index symbol will occur in the defining expression, regardless of the frequency of its occurrence in that expression. With these conventions understood, define

$$A_n=D_m r^{m+n-1}\ , \tag{6.1}$$

where, as before,

$$D_m=\frac{(-1)^m}{c^m m!}\frac{d^m}{dt^m}\ . \tag{6.2}$$

Then there immediately follows

$$-\frac{1}{c}DD_m=(m+1)D_{m+1}\ , \tag{6.2'}$$

where $D=d/dt$, as before.

Also, in a straightforward manner, one may show that

$$-\frac{1}{c}DA_n=D_m(mr^{m+n-2})\ . \tag{6.3}$$

It will be useful to perform binomial expansions of certain expressions as follows: given $f(m)$ and g as arbitrary scalar or, where appropriate, vector functions, then

$$\begin{aligned}
D_m(f(m)g)&=\sum_{m=0}^{\infty}\frac{(-1)^m}{c^m m!}D^m(f(m)g)\\
&=\sum_{m=0}^{\infty}\frac{(-1)^m}{c^m m!}\sum_{n=0}^{m}\begin{bmatrix}m\\ n\end{bmatrix}D^{m-n}f(m)D^n g\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(-1)^{m-n}}{c^{m-n}(m-n)!}D^{m-n}f(m)\\
&\qquad\times\frac{(-1)^n}{c^n n!}D^n g\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{m}D_{m-n}f(m)D_n g\\
&=\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}D_{m-n}f(m)D_n g\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}D_m f(m+n)D_n g\\
&=D_m f(m+n)D_n g\ .
\end{aligned} \tag{6.4}$$

In particular, if $f(m)=r^{m+l-1}$, then

$$D_m(gr^{m+l-1})=D_m r^{m+n+l-1}D_n g=A_{n+l}D_n g\ , \tag{6.5}$$

that is, in Eq. (6.4) or (6.5), the expressions on the left may be expanded in a series of terms which are linearly dependent upon the derivatives of g.

In the above notation, using Eq. (6.5), formulas (4.3) and (4.4) become

$$V(t)=\frac{q}{4\pi\epsilon_0}A_0=\frac{q}{4\pi\epsilon_0}D_m r^{m-1} \tag{6.6}$$

and

$$\mathbf{A}(t)=\frac{\mu_0 q}{4\pi}A_n D_n \mathbf{v}_p\ , \tag{6.7}$$

respectively.

Note that for uniform source velocity, the standard result,

$$\mathbf{A}=\frac{\mu_0 q}{4\pi}A_0\mathbf{v}_p=\frac{1}{c^2}V\mathbf{v}_p\ , \tag{6.8}$$

follows immediately.

B. The fields

Linear expansions of the electric and magnetic fields may be developed directly from Eqs. (6.6) and (6.7) as follows:

$$\frac{4\pi\epsilon_0}{q}\nabla_F V=\nabla_F A_0$$
$$=D_m[(m-1)r^{m-3}\mathbf{r}]$$
$$=D_m[(m+n-1)r^{m+n-3}]D_n\mathbf{r}\ , \tag{6.9}$$

using Eq. (6.4); and

$$\frac{4\pi\epsilon_0}{q}\frac{\partial \mathbf{A}}{\partial t}=\frac{1}{c^2}D(A_n D_n \mathbf{v}_p)=-\frac{1}{c^2}D(A_n D D_n \mathbf{r})$$
$$=\frac{1}{c}D[A_n(n+1)D_{n+1}\mathbf{r}]\ \ [\text{by } (6.2')]$$
$$=-D_m(mr^{m+n-2})(n+1)D_{n+1}\mathbf{r}-A_n(n+1)(n+2)D_{n+2}\mathbf{r}\ \ [\text{by } (6.3) \text{ and } (6.2')]$$
$$=-D_m(mnr^{m+n-3})D_n\mathbf{r}-A_{n-2}(n-1)nD_n\mathbf{r}$$
$$=-D_m[(m+n-1)nr^{m+n-3}]D_n\mathbf{r}\ \ [\text{by } (6.1)]\ . \tag{6.10}$$

Therefore

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}E_n D_n\mathbf{r}\ , \tag{6.11}$$

where

$$E_n=(n-1)D_m[(m+n-1)r^{m+n-3}]\ . \tag{6.12}$$

Similarly,

$$\frac{4\pi}{\mu_0 q}\mathbf{B}=\frac{4\pi}{\mu_0 q}\nabla_F\times\mathbf{A}=\nabla_F\times(A_n D_n\mathbf{v}_p)$$
$$=(\nabla_F A_n)\times D_n\mathbf{v}_p$$
$$=D_m[(m+n-1)r^{m+n-3}\mathbf{r}]\times D_n\mathbf{v}_p$$
$$=D_m[(m+n+k-1)r^{m+n+k-3}]D_k\mathbf{r}\times D_n\mathbf{v}_p\ \ [\text{by } (6.4)]\ .$$

Therefore

$$\mathbf{B}=\frac{\mu_0 q}{4\pi}B_{k+n}D_k\mathbf{r}\times D_n\mathbf{v}_p\ , \tag{6.13}$$

where

$$B_n=D_m[(m+n-1)r^{m+n-3}]\ . \tag{6.14}$$

It is readily apparent from Eqs. (6.12) and (6.14) that

$$E_n=(n-1)B_n\ . \tag{6.15}$$

Equations (6.11) and (6.13) are the complete linear expansions of the electric field and magnetic induction in terms of the vector components of the source motion kinematics.

Equation (6.11) is interesting in that it has no pure velocity component ($n=1$) under any circumstances. In particular, for uniform velocity, it reduces to

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}D_m[(1-m)r^{m-3}]\mathbf{r}\ . \tag{6.16}$$

The pure radial (though nonradially symmetric) character of the **E** field is also apparent from Eq. (6.16).

The coefficient of **r** in Eq. (6.16) may be computed exactly for the case of uniform source motion. For uniform motion along the x axis and a field point in the $x=0$ plane a distance ρ from the x axis, procedures similar to those detailed in Sec. V yield

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{[\rho^2(1-\beta^2)+v^2t^2]^{3/2}}\mathbf{r}\ , \tag{6.17}$$

in agreement with the standard result.

The complex structure of the magnetic field—in terms of underlying source particle kinematics—is evident from Eq. (6.13). Every possible cross product combination is represented. In the case of uniform source particle motion, Eq. (6.13) reduces to the (modified) Biot-Savart Law,

$$\mathbf{B}=\frac{\mu_0 q}{4\pi}D_m[(1-m)r^{m-3}]\mathbf{v}_p\times\mathbf{r} \tag{6.18}$$

or, using Eq. (6.16),

$$\mathbf{B}=\frac{1}{c^2}\mathbf{v}_p\times\mathbf{E}\ , \tag{6.19}$$

again, in agreement with the standard result.

The expression for the magnetic induction may be presented in factored form using Eq.(4.27). Only $[\hat{\mathbf{n}}]'$ requires evaluation in that case, for which another method of developing expansions will be introduced.

Noting that $[\mathbf{r}]'=\mathbf{r}([t]')$ and that $[t]'=t-[r]'/c$, where primed brackets indicate retarded-time evaluations,

$$[\mathbf{r}]'=\mathbf{r}(t-[r]'/c)$$
$$=[r]'^{\,n}D_n\mathbf{r}\quad\text{(Taylor series expansion)}\,, \tag{6.20}$$

so that

$$[\hat{\mathbf{n}}]'=[r^{n-1}]'D_n\mathbf{r}$$
$$=D_m\left[\left[1+\frac{\dot r}{c}\right]r^{m+n-1}\right]D_n\mathbf{r}\quad[\text{by }(4.12)]$$
$$=N_nD_n\mathbf{r}\,, \tag{6.21}$$

where

$$N_n=(n-1)D_m\frac{r^{m+n-1}}{m+n-1}\,. \tag{6.22}$$

The proof of Eq. (6.22) is as follows:

$$D_m\left[\left[1+\frac{\dot r}{c}\right]r^{m+n-1}\right]=D_mr^{m+n-1}+\frac{1}{c}DD_m\frac{r^{m+n}}{m+n}$$
$$=D_mr^{m+n-1}-(m+1)D_{m+1}\frac{r^{m+n}}{m+n}\quad[\text{by }(6.2')]$$
$$=D_mr^{m+n-1}-mD_m\frac{r^{m+n-1}}{m+n-1}$$
$$=N_n\,.$$

For $n=0$, $m=1$ or $n=1$, $m=0$ Eq. (6.22) yields correct results if the value for m is assigned first, followed by the value for n, so that

$$N_0=\frac{1}{r}+\frac{1}{r}\frac{\dot r}{c}+\cdots\quad\text{and}\quad N_1=1\,. \tag{6.23}$$

The magnetic induction, in factored form, is then given by

$$\mathbf{B}=\frac{1}{c}[\hat{\mathbf{n}}]'\times\mathbf{E}$$
$$=\frac{q}{4\pi\epsilon_0c}(N_mD_m\mathbf{r})\times(E_nD_n\mathbf{r})\,. \tag{6.24}$$

Yet another form [see Eq. (4.7)] for the magnetic induction is possible if, in Eq. (6.13), the subscripts k and n are absorbed using Eq. (6.4), so that

$$\mathbf{B}=\frac{\mu_0q}{4\pi}D_m[(m-1)r^{m-3}\mathbf{r}\times\mathbf{v}_p]\,. \tag{6.25}$$

In the expansion of Eq. (6.25), the lowest-order (in $1/c$) term is clearly the classical Biot-Savart law, and the first-order term is missing, just as it is for the electric field expansion, Eq. (6.11).

Finally, using Eq. (6.4), Eq. (6.25) may be fully expanded to yield

$$\mathbf{B}=\frac{\mu_0q}{4\pi}D_m[(m+n-1)r^{m+n-3}]D_n(\mathbf{r}\times\mathbf{v}_p)$$
$$=\frac{\mu_0q}{4\pi}B_nD_n(\mathbf{r}\times\mathbf{v}_p)\,, \tag{6.26}$$

a form which could have been derived directly from Eq. (6.13) using the binomial expansion to collect derivatives.

The intimate relationship of the electric field to the motion of the source and the magnetic field to the angular motion of the source (relative to the field point) is clearly displayed and distinguished in Eqs. (6.11) and (6.26).

That there can be notational advantages gained in retaining retarded-time representations is shown by evaluating $\mathbf{B}$ for a uniform velocity source particle. In this case, the Taylor expansion for $[\hat{\mathbf{n}}]'$ [Eq. (6.20)] yields

$$[\hat{\mathbf{n}}]'=\frac{\mathbf{r}}{[r]'}+\frac{\mathbf{v}_p}{c}\,, \tag{6.27}$$

while $\mathbf{E}$ is given by Eq. (6.17), so that

$$\mathbf{B}=\frac{1}{c}[\hat{\mathbf{n}}]'\times\mathbf{E}=\frac{\mu_0q}{4\pi}\frac{1-\beta^2}{[\rho^2(1-\beta^2)+v^2t^2]^{3/2}}\mathbf{v}_p\times\mathbf{r}\,, \tag{6.28}$$

again, in agreement with the standard results. Note that the last result also follows—trivially—from Eqs. (6.17) and (6.19).

VII. ADVANCED POTENTIALS

It is well known [11] that the complete mathematical solution of any inhomogeneous wave equation is composed of both retarded- and advanced-time functional representations. Retarded-time representations have been discussed up to this point, whereas advanced-time representations have not. An advanced-time functional representation may be formally defined as any function evaluated at an advanced time $[t]''$, determined by the equation

$$[t]''=t+r([t]'')/c\,, \tag{7.1}$$

where

$$r([t]'')=|\mathbf{r}_F-\mathbf{r}_p([t]'')|\,. \tag{7.2}$$

The use of advanced-time solutions to wave equations admits no simple physical interpretation, since, with $r>0$, Eq. (7.1) shows that $[t]''>t$. Therefore $[t]''$ is some future time with respect to the present time t, and one's customary sense of causality—cause preceding effect—is violated. Despite the failure to comfortably interpret the meaning of such solutions, they nevertheless exist mathematically, and one becomes hard pressed to exclude them based upon interpretive failure alone. Interpretive failure becomes all the more moot, in any case, when both retarded and advanced solutions are cast into a present-time formulation, for then only the present-time state of the source particle motion is involved, and the need for a physical interpretation (involving past or future events) is somewhat obviated. Therefore it would seem prudent to include both types of solutions in the construction of fields, while relying on theoretical and experimental consequences to discriminate the mix.

It is felt by some (for example, Panofsky and Phillips [11]) that only the retarded solutions are experientially valid in a distant field approximation, but that both solutions may find applicability otherwise. Others argue that the advanced-time solutions have no physical or logical underpinning whatsoever and that one may therefore discard them altogether. Such a conclusion is explicit or implicit in the program of many undergraduate and graduate textbooks in common use (for example, Jackson [10], Morse and Feshbach [12], Corson and Lorrain [13], Smythe [14]). Others (for example, Stratton [15]) caution against the application of "logical" causality principles in discarding advanced-time solutions but nevertheless develop an exclusively retarded action theory based on its presumed conformity to physical data. This seems wise but raises the question of whether a mix of retarded- and advanced-time solutions could not conform to physical data as well as the retarded solutions alone. This paper makes no preemptive judgment, one way or the other. It supports (along with Synge [16]) the inclusion of both solutions since both are mathematically present. That they may not be freely joined in linear combination, however, will be shown below.

It will be found in the treatment that follows that the experimental measurement of a single parameter, called the (causality) *mix* parameter, can decide the issue of solution inclusion for charged particles in motion. That is, the experimental measurement of a single number can decide in favor of retarded solutions, advanced solutions, or a mixture of both solutions. This parameter will be developed after first describing how to obtain advanced-time solutions from the material thus far presented.

A. Advanced-time formulations

Noting that the only mathematical difference between Eqs. (7.1) and (7.2) above and the corresponding equations in Sec. II is the substitution of $-c$ for c, it follows that all of the results of the preceding sections, with $-c$ replacing c, should yield all of the relevant advanced-time field representations. That this is true is assured by noting that the validity of Eq. (3.2) is independent of the algebraic sign of σ. That is, a contour which yields a convergent retarded-time Lagrange series by virtue of the satisfaction of Eq. (3.1) along the contour will also yield, along the same contour, a convergent advanced-time Lagrange series with the substitution of $-c$ for c in that series. Thus almost everything that has been developed up to this point applies equally well to advanced-time formulations, with only a change in sign of the speed-of-light parameter c.

B. Mixed potentials and the causality parameter

Assuming V_r and $\mathbf{A}_r$ to be retarded potentials and V_a and $\mathbf{A}_a$ to be advanced potentials, there follows that if α_1 and α_2 are scalar parameters for which

$$V=\alpha_1 V_r+\alpha_2 V_a \ , \tag{7.3}$$

then

$$\Box_F^2 V=\alpha_1\Box_F^2 V_r+\alpha_2\Box_F^2 V_a=-\frac{(\alpha_1+\alpha_2)\rho}{\epsilon_0}=-\frac{\rho}{\epsilon_0}$$

provided

$$\alpha_1+\alpha_2=1 \ . \tag{7.4}$$

That is, if Eq. (7.4) is satisfied, then the potential defined by Eq. (7.3) satisfies the same inhomogeneous wave equation that is satisfied by the individual retarded and advanced potentials.

If now the Lorentz gauge condition is to be satisfied, namely,

$$\nabla_F\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}=0 \ , \tag{7.5}$$

then it is easy to demonstrate that an associated vector potential, comprised of a linear combination of retarded and advanced components, must be given by

$$\mathbf{A}=\alpha_1\mathbf{A}_r+\alpha_2\mathbf{A}_a \ . \tag{7.6}$$

Assuming Eq. (7.4) and (having already adopted) the Lorentz gauge condition, it follows that the scalar potential defined by Eqs. (7.3) and (7.4) represent the most general solution of the inhomogeneous wave equation for point sources. But $V_a(c)=V_r(-c)$ by the discussion in Sec. III A, so that, using Eqs. (6.6), there follows (retaining the summation convention on repeated indices), for a point charged particle,

$$\begin{aligned} V&=\frac{q}{4\pi\epsilon_0}\frac{\alpha_1(-1)^m+\alpha_2}{c^m m!}D^m r^{m-1} \\ &=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{c^{2m}(2m)!}D^{2m}r^{2m-1}\right. \\ &\qquad\left.+\alpha\frac{1}{c^{2m+1}(2m+1)!}D^{2m+1}r^{2m}\right] \\ &=V_D+\alpha V_M \ , \end{aligned} \tag{7.7}$$

where

$$\alpha=\alpha_2-\alpha_1 \tag{7.8}$$

is some undetermined parameter of the general expansion.

The partial potential expansions,

$$V_D = \frac{q}{4\pi\epsilon_0}\frac{1}{c^{2m}(2m)!}D^{2m}r^{2m-1} \tag{7.9a}$$

and

$$V_M = \frac{q}{4\pi\epsilon_0}\frac{1}{c^{2m+1}(2m+1)!}D^{2m+1}r^{2m} , \tag{7.9b}$$

will be referred to as *dominant* and *minor* scalar potentials, for expository convenience. The terminology is also suggestive of the relative importance of the two series at particle velocities much less than the speed of light.

In a similar way, Eq. (7.6) represents the most general solution (under Lorentz gauge coupling) of the inhomogeneous wave equation for the vector potential. But $\mathbf{A}_a(c) = \mathbf{A}_r(-c)$, so that, if Eq. (4.4) is used, there follows

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 q}{4\pi}\frac{\alpha_1(-1)^m+\alpha_2}{c^m m!}D^m(\mathbf{v}_p r^{m-1}) \\ &= \frac{\mu_0 q}{4\pi}\left[\frac{1}{c^{2m}(2m)!}D^{2m}(\mathbf{v}_p r^{2m-1}) + \alpha\frac{1}{c^{2m+1}(2m+1)!}D^{2m+1}(\mathbf{v}_p r^{2m})\right] \\ &= \mathbf{A}_D + \alpha\,\mathbf{A}_M \ .\end{aligned} \tag{7.10}$$

The partial potential expansions,

$$\mathbf{A}_D = \frac{\mu_0 q}{4\pi}\frac{1}{c^{2m}(2m)!}D^{2m}(\mathbf{v}_p r^{2m-1}) \tag{7.11a}$$

and

$$\mathbf{A}_M = \frac{\mu_0 q}{4\pi}\frac{1}{c^{2m+1}(2m+1)!}D^{2m+1}(\mathbf{v}_p r^{2m}) , \tag{7.11b}$$

may similarly be referred to as *dominant* and *minor* vector potentials.

The parameter α may be appropriately termed a (causality) *mix* parameter since, if it assumes a value in the range of values $[-1,1]$, it determines the mix or "weight" accorded the retarded and advanced potential solutions in forming the composite solution for the potentials. For example, if $\alpha=-1$, the potentials consist entirely of the retarded-time component, with no advanced-time component. The situation is reversed if $\alpha=1$. If $\alpha=0$, equal "weights" of both solutions occur. If α falls outside the range $[-1,1]$, then at least one of the set $\{\alpha_1,\alpha_2\}$ is negative, and the simple "weighted mix" interpretation is lost.

The potential of Eq. (7.7) may be termed a *mixed* potential by reason of the presence of the parameter α. It is apparent that, unless this parameter's value is fixed in nature, the potential of a charged particle may not be unique. It is also clear that a claim that advanced potentials are physically or logically untenable—and are therefore to be discarded—is equivalent to the assignment $\alpha=-1$. But does nature permit this assignment and hence the claim? It might be based upon both an anthropomorphic need for "common sense" understanding and justification, and a pragmatic need to assign a unique potential to charged particle processes for the sake of intellectual progress. The central question, however, begged by the presence of the parameter α in Eqs. (7.7) or (7.10), is really this: can speculation as to its nature and value be replaced with experimentation?

Noting that the first nonvanishing term of the α bearing (minor) series is of order $1/c^3$, one reasonably concludes that a precise measurement of charged particle fields at relativistic speeds would necessarily be involved. However, speeds near the speed of light are not enough. Significant accelerations must also be provided. To see this, recall from the considerations leading up to Eqs. (5.12) that all odd derivative terms of the Lagrange series expansion for the scalar potential vanished under arbitrary constant velocity source particle motion. These are precisely the terms comprising the minor potential series given in Eqs. (7.7) and (7.9b). The minor series multiplying α in Eq. (7.7) therefore always vanishes for constant velocity source particle motion, and no constant source particle velocity experiment can then reveal the magnitude of α. [α cannot, of course, be determined from the vector potential expansion either, by virtue of Eq. (5.16).] Thus acceleration is required. It must be large, as must be the velocity, in order to overcome the $1/c^3$ dominance—else the experiment must be reasonably precise.

Given that experiments can be performed to estimate the value of α, they should be sufficiently varied to discriminate whether α is fixed or, in some fashion, dependent upon experimental conditions. If the latter case is true, then charged particle electromagnetics (and consequently, electrodynamics) may suffer with respect to the present (Maxwellian) theory of field generation, since the uniqueness of the elementary potential expansions would then be lost. If, on the other hand, it results that the α is experimentally fixed, then the true nature of causality (and perhaps other processes) will have been revealed (though, perhaps, still not fully understood), and Maxwellian theory will have been augmented with another empirical fact.

No specific experimental procedure will be addressed in this paper, because of its limited scope. It appears worth the investment, however, in the design of any experiment, to attempt a quite rigorous—if not exact—evaluation of the dominant and minor series expansions presented in Eqs. (7.9) and (7.11) and then to derive the electric and magnetic fields (where appropriate) from the potentials so determined. A reasonably exact experimental procedure from that point on could well reveal the true nature of causality in the charged particle regime.

If any argument is to be advanced concerning the value of α, assuming it is a fixed value, then one would be hard pressed to argue any value other than $\alpha=0$, since the assumed complete knowledge of the present-time state of the source particle motion weights equally in terms of knowledge of its past and present kinematic behavior. If a fixed value for α were to be anticipated, then $\alpha=0$ appears to be a viable candidate, *even though elementary causality is violated,* because past and future events are seen here to equally affect the present event.

The case for $\alpha=0$ is additionally bolstered by the fact that $V=V_D$ for constant velocity charged particles. If $\alpha\neq 0$, the transition between two different states of uniform velocity via an intermediate state of acceleration results in a type of discontinuity in functional form; that is, $V_D \rightarrow V_D+\alpha V_M \rightarrow V_D$ during the transition. Though no known law is violated in this process, there is a sense of intrinsic continuity which is nevertheless violated. Argumentation along this line is difficult to pursue, however, and it may in the end result that there is nothing whatsoever wrong with such process representation.

C. Some open questions concerning potentials and causality

All of the preceding discussion rests wholly upon the assumption that, in electromagnetic interactions, the $1/r$ term must appear in all potential expansions. It does so uniquely if advanced potentials are discarded; that is, if one of the linearly independent solutions is selectively ignored. If both linearly independent solutions are initially accorded equal importance, however, then the potential representation

$$V=\alpha_1 V_D+\alpha_2 V_M \tag{7.12}$$

is a more satisfactory resolution of the potential into linearly independent terms, one of which carries the $1/r$ potential dependence term, the other of which does not. It is more satisfactory because V_D contains *all* of the $1/r$ dependence, prior to making any assumptions. When advanced potentials are discarded, for example, *some* $1/r$ dependence is discarded as well.

Now if the claim holds that the $1/r$ term must persist in electromagnetic interactions, then the assignment $\alpha_1=1$, $\alpha_2=\alpha$ returns to the preceding discussion, along with the argument that $\alpha=0$. That is, that V_D is the correct form of the potential when $1/r$ dependence must be present. But it may be argued that if one linearly independent component of the potential has physical significance, then so should the other. That is, unless strong experimental or theoretical prohibitions exist, one must look at natural phenomena for the independent occurrence of the potential V_M as well. Can electromagnetic interaction occur without a $1/r$ potential dependence? If so, is it always so, or does there occur a transition from V_D dependence to V_M dependence, for example, when moving from certain free states to certain bound states? Obviously, these questions cannot be answered easily and will rely heavily upon experimental or further theoretical investigation.

Let sight not be lost of the fact that all undertakings in this area of investigation directly impact the concept of causality and one's understanding of it. Should it result that $\alpha\neq -1$, then one's conventional understanding of the universe must suffer a reconstruction whose end is far from clear. In particular, if it results that $\alpha=0$, then the linearly independent solution V_M must be sought in nature, or an understanding of why it does not occur must be sought. If $\alpha\neq 0$ and $\alpha\neq -1$, then even greater explanatory difficulties will exist and the very equations which generated these solutions would likely be called into question. It might result, for example, that the second-order character of the wave equation is fatally insufficient to guarantee a unique solution for the potentials and an acceptable first-order theory must then be sought; or inaccuracies may exist in the mathematical representation of the source function. In any case, which difficulty will manifest itself and which set of answers must be sought must properly await the impartial judgment of experimentation.

Until a resolution of the above questions is achieved, it should be borne in mind that all of the field formulas developed in preceding sections are valid for conventional (retarded-time) causality only. Should $\alpha\neq -1$, then all of the preceding formulas must be recast in the manner of the mixed potentials presented in this section. *Without* an experimental knowledge of α or compelling theoretical considerations, it is the judgment of this paper that the exact nature of the electromagnetic (and other wave-theoretic) potentials must be considered unknown.

VIII. SUMMARY

The Introduction to this paper addresses the relationship of this paper's contents to the modern wave-particle duality concept for field quanta and clarifies the close relationship that exists between solutions of Poisson's and d'Alembert's equations with respect to action-at-a-distance and propagating wave solution representations. The paper also constructs, in terms of a Lagrange series expansion, an analytic solution of the retarded-time equation, Eq. (2.1), by utilizing an inversion theorem based upon analytic function theory, whose proof is presented in Sec. III. The theorem allows the recasting of the Liénard-Wiechert potentials from their retarded-time representation to a present-time or instantaneous action-at-a-distance representation, as given by Eqs. (4.3) and (4.4). By simple extension, instantaneous action-at-a-distance solutions of the inhomogeneous wave equation with point sources, Eq. (1.8), are shown to be given in all cases by Eq. (1.11). Electric and magnetic fields are also developed in an instantaneous action-at-a-distance format in Sec. IV and are shown to be identical to classical retarded-time formulations for the fields. Some examples which validate the resulting equations are presented in Sec. V. Also presented in Sec. V is a demonstration of the relativistic correctness of the paper's field formulations. Section VI shows that by Lagrange series manipulation, the electromagnetic fields and potentials can be cast into various forms that may be useful for theoretical, mathematical, and applied investigations. Section VII introduces and discusses advanced potentials and their instantaneous action-at-a-distance representations. Their significance to field theories is addressed and the possibility is raised of performing experiments to determine if advanced (electromagnetic) potentials have phenomenological reality. The necessity of performing such experiments is also discussed, if the uniqueness of electromagnetic and other wave-theoretic potentials is to be unambiguously established. Equations necessary to perform such experimentation are presented, but not reduced to explicit experimental detail. Throughout, only the topic of fields generated from the arbitrary motion of point source particles is addressed.

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APPENDIX

1. Representations of ρ' in Eq. (1.6)

If the field point $\mathbf{r}_F$ and source point $\mathbf{r}_s$ are fixed, then, with $r=|\mathbf{r}_F-\mathbf{r}_s|$,

$$\rho(t')=\rho(\mathbf{r}_s,t-r/c) \quad [\text{by } (1.5)]$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n r^n}{c^n n!}\frac{d^n}{dt^n}\rho(\mathbf{r}_s,t)=\rho'(t) \quad (\text{Taylor series})$$

and

$$\phi=-G\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}\frac{d^n}{dt^n}R_n(\mathbf{r}_F,t) \quad [\text{by } (1.4)] ,$$

where

$$R_n(\mathbf{r}_F,t)=\int\rho(\mathbf{r}_s,t)r^{n-1}dv_s .$$

For particulate sources, where

$$\rho(\mathbf{r}_s,t)=m\delta(\mathbf{r}_s-\mathbf{r}_p(t)) ,$$

there results

$$R_n(\mathbf{r}_F,t)=m|\mathbf{r}_F-\mathbf{r}_p(t)|^{n-1} ,$$

$$\phi=-mG\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}\frac{d^n}{dt^n}|\mathbf{r}_F-\mathbf{r}_p(t)|^{n-1} \quad [\text{by } (1.4)] ,$$

and

$$\rho'(t)=m\sum_{n=0}^{\infty}\frac{(-1)^n r^n}{c^n n!}\frac{d^n}{dt^n}\delta(\mathbf{r}_s-\mathbf{r}_p(t)) .$$

2. Proof of Eq. (4.14)

$$D_n(n\mathbf{g}r^{n-1})=D_{n,1}(n\mathbf{g}r^{n-1})$$
$$=-\frac{1}{c}D_{n-1,1}D(\mathbf{g}r^{n-1})$$
$$=-\frac{1}{c}D_nD(\mathbf{g}r^{n})$$
$$=-\frac{1}{c}DD_n(\mathbf{g}r^{n})$$
$$=-\frac{1}{c}D\left[\frac{\mathbf{g}}{\chi}\right]' \quad [\text{by } (4.12)]$$
$$=-\frac{1}{c}\left[\frac{1}{\chi}D\right]'\left[\frac{\mathbf{g}}{\chi}\right]' \quad [\text{by } (4.16)]$$
$$=-\frac{1}{c}\left[\frac{1}{\chi}D\left[\frac{\mathbf{g}}{\chi}\right]\right]'$$
$$=-\frac{1}{c}D_n\left[D\left[\frac{\mathbf{g}}{\chi}\right]r^n\right] \quad [\text{by } (4.12)] .$$

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